# Double stochastic resonance over an asymmetric barrier

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In a recent experiment [Müller *et al.*, Phys. Rev. A **79**, 031804(R) (2009)] reported a splitting of the stochastic resonance peak, which they attributed to the asymmetry of an effective double-well restoring potential in their optomechanical read-out device. We show here that such an effect, though smaller than reported, is indeed consistent with a characterization of stochastic resonance as a synchronization phenomenon, while it proves elusive in terms of spectral quantifiers.

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## I. INTRODUCTION

Stochastic resonance [1,2] is by now a textbook example of how noise can best enhance the response of a bistable system to an external periodic drive. Historically, the research on stochastic resonance (SR) focused on symmetric bistable systems, either continuous or discrete [3–5]. Asymmetry effects have also been taken into account, mostly by tilting an otherwise symmetric double-well potential [6,7]. This led to the appearance of deltalike spikes for both odd *and* even multiples of the drive frequency in the power spectral density (p.s.d.) of the system output. The spikes of all harmonics were found to go through a maximum as a function of the noise intensity and the tilt. In particular, when plotted versus the noise intensity, the strength of the first and most prominent spike, corresponding to the drive frequency, was reported to exhibit a characteristic *single* broad peak, termed SR peak; asymmetry was observed to degrade the magnitude of the SR signature, without affecting the overall picture [5]. An analysis in terms of residence time distributions [8] failed to add insight to the above picture: The asymmetry of the confining potential was deemed solely responsible for the onset of the even spectral spikes in the output p.s.d. of systems undergoing SR [1].

In a recent experiment, Mueller and co-workers [9,10] investigated an optomechanical torsion oscillator confined to two skewed (asymmetric) stable states. They characterized the system response to an external ac drive in terms of a synchronization quantifier, named degree of coherence and denoted here $C$, which is defined as the ratio of the total number of interwell transitions around the signal half period, to the total number of possible transitions. Contrary to the case of SR in symmetric bistable systems [1,11], $C$ was observed to develop two distinct maxima when plotted versus the noise intensity. Mueller and co-workers attributed such a "SR splitting" to the interplay of the unavoidable interwell and intrawell asymmetries built in the restoring potential of their torsion oscillator.

In this Brief Report, we numerically investigate SR in an asymmetric double-well (ADW) potential obtained by deforming a symmetric one, so that the energy barrier opposing transitions to the right and to the left stays the same. Numerical simulation allows a better control on the potential shape and a direct quantitative comparison between synchronization and spectral manifestations of the SR phenomenon. We conclude that double SR (the "splitting SR" reported in [9]) is a peculiar, though relatively small, synchronization effect without a sizable spectral counterpart.

## II. MODEL

In order to focus on the role of the sole *intrawell* asymmetry, which is on the different skewness of two competing binding wells, we adopted the following deformable ADW potential [12],

$$V(x) = V_0\left(\frac{\{1-\exp[-b_+(x+1)]\}\{1-\exp[b_-(x-1)]\}}{\{1-\exp[-b_+(x_0+1)]\}\{1-\exp[b_-(x_0-1)]\}}\right)^2, \quad (1)$$

with potential minima set at $x=\pm 1$. The parameters $x_0$ and $V_0$ have been introduced for convenience, respectively to denote the position of the barrier and its height, $\Delta V$. The tunable parameters $b_\pm$ control, besides $x_0$, the potential curvature both at the bottom of wells, $\omega_\pm^2 = V''(\pm 1)$, and at the top of the barrier, $\omega_0^2 = |V''(x_0)|$. The ADW potential Eq. (1) used in our simulations is sketched in Fig. 1. Note that the energy barrier $\Delta V$ is the same for both transitions $\mp \to \pm$, i.e., according to the notation of [9], we suppressed the *interwell* asymmetry. Moreover, in order to enhance the effect of the intrawell asymmetry, we set the curvature of the negative well far smaller than that of the positive well, $\omega_- < \omega_+$.

An overdamped Brownian particle diffusing in the potential Eq. (1), subjected to a sinusoidal drive of amplitude $A$ and angular frequency $\Omega$, is described by the Langevin equation,

$$\dot{x} = -V'(x) + A\cos(\Omega t - \phi) + \xi(t), \quad (2)$$

with $A \geq 0$ and $\phi$ an arbitrary constant. Here, $\xi(t)$ denotes a Gaussian white noise with zero mean and intensity $D$, that is $\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t')$.

The dynamics of Eq. (2) was numerically simulated by means of a standard algorithm for the integration of stochastic differential equations. The residence time distributions

 

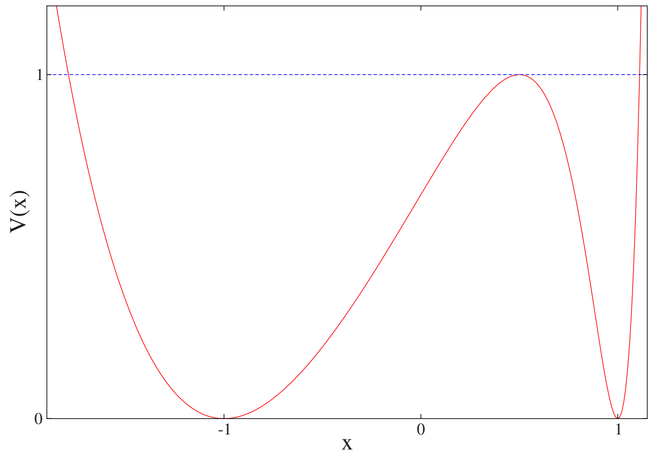


FIG. 1. (Color online) Our model ADW potential Eq. (1) for $b_+=0.5$, $b_-=5$ and $V_0=1$. The barrier turns out to be centered at $x_0 \simeq 0.5$ and $\Delta V=1$ high. The curvatures of the potential at $x=\pm 1$ and $x_0$ are, respectively, $\omega_-^2=2.1$, $\omega_+^2=85.2$, and $\omega_0^2=5.7$.

and the spectral properties of the stochastic process $x(t)$ could be determined with high numerical accuracy and extremely good statistics [1].

## III. DOUBLE STOCHASTIC RESONANCE

Following the approach of [9], we determined the normalized distributions of the residence times $T$ in the positive, $N_+(T)$, and in the negative potential wells, $N_-(T)$. Correspondingly, one can interpret $N_\pm(T)$ as the distributions of the waiting times, respectively, for the right-to-left and left-to-right transitions. According to the so-called *bona fide* SR scheme [11], such distributions peak at the odd multiples of the half-driving period, $T_n=(n-\frac{1}{2})T_\Omega$, with $n$ an integer, and $T_\Omega=2\pi/\Omega$. More importantly, under the synchronization condition [1,11]

$$T_\Omega = 2T_K^\pm, \tag{3}$$

the main SR peak of $N_\pm(T)$, i.e., the peak with $n=1$, dominates both over all higher order peaks, with $n>1$, and over the exponential background of the random (or incoherent) switches. Here, $T_K^\pm$ denote the average waiting times for the unbiased, $A=0$, random switches to the left and to the right, respectively. For relatively small noise intensities, $D \lesssim \Delta V$, these two escape times can be safely approximated by means of Kramers' formula [13],

$$\frac{1}{T_K^\pm} = \frac{\omega_\pm \omega_0}{2\pi} \exp\left(-\frac{\Delta V}{D}\right). \tag{4}$$

The peak structures of $N_\pm(T)$ in Fig. 2 display a clearly distinct $D$ dependence: The SR synchronization condition for $N_-(T)$ tends to occur at values of $D$ higher than for $N_+(T)$. We have, thus, recovered numerically the asymmetric synchronization mechanism earlier reported by Mueller and coworkers [9].

For a more quantitative analysis of such a mechanism, we introduced the following definition of degree of coherence, namely,

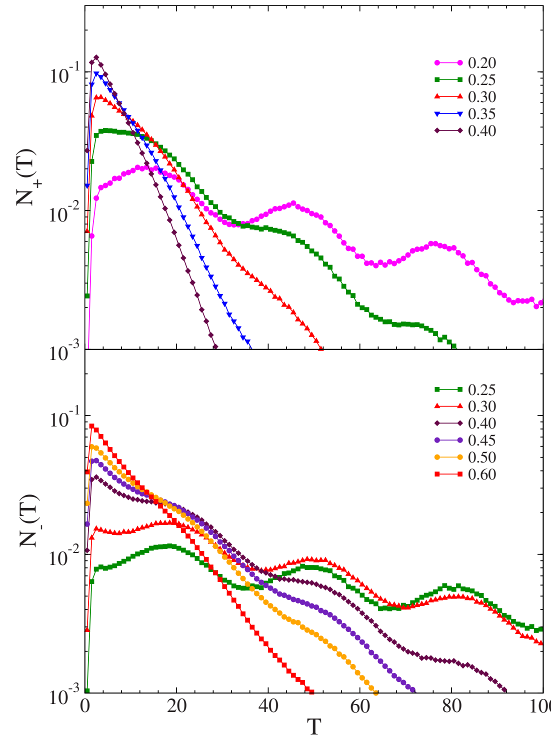


FIG. 2. (Color online) Residence time distributions $N_\pm(T)$ in the ADW potential Eq. (1) for different values of $D$ (in the legends). The subscripts $\pm$ denote the transitions, respectively, right-to-left and left-to-right (see text). Other simulation parameters: $A=0.2$, $\Omega=0.2$ and the ADW parameters as in Fig. 1.

$$C_\pm = \int_{\frac{1}{2}(T_\Omega-\Delta t)}^{\frac{1}{2}(T_\Omega+\Delta t)} N_\pm(T)dT. \tag{5}$$

$C_\pm$ denotes the fraction of the waiting times (in the $x=\pm 1$ well) that fall within a time interval (or bin) $\Delta t$ centered around the half-driving period, $T_\Omega/2$. $C_\pm$ can be regarded as a measure of the magnitude of the first $N_\pm(T)$ peak. In the main panel of Fig. 3 we display $C_\pm$ versus $D$ for different drive amplitudes and the same small ratio $\Delta t/T_\Omega$ as in [9]. Moreover, for the sake of a comparison with [9], we also plotted the average $\bar{C}=\frac{1}{2}(C_+ + C_-)$, which coincides with the coherence quantifier, $C$, measured experimentally by Mueller and co-workers.

The effects of intrawell asymmetry on the SR mechanism taking place in the ADW potential Eq. (1) are now apparent: (i) The curves $C_\pm$ peak at different noise intensities, $D_\pm$, with $D_+<D_-$; (ii) The curves $\bar{C}$ exhibit a prominent peak in correspondence of the $C_+$ maxima and a hardly detectable side bump in correspondence of the $C_-$ maxima. The curves $\bar{C}(D)$ are our numerical counterpart of the plot shown in Fig. 4(d) of [9]; (iii) The position, $D_\pm$, of the $C_\pm$ peaks are consistent with the synchronization condition (3). In particular, we checked numerically that

$$\frac{D_- - D_+}{D_+ D_-} \simeq \frac{1}{\Delta V} \ln\left(\frac{\omega_+}{\omega_-}\right), \tag{6}$$

as one would expect on combining Eqs. (3) and (4). The approximate equality Eq. (6) only applies for ADW poten-

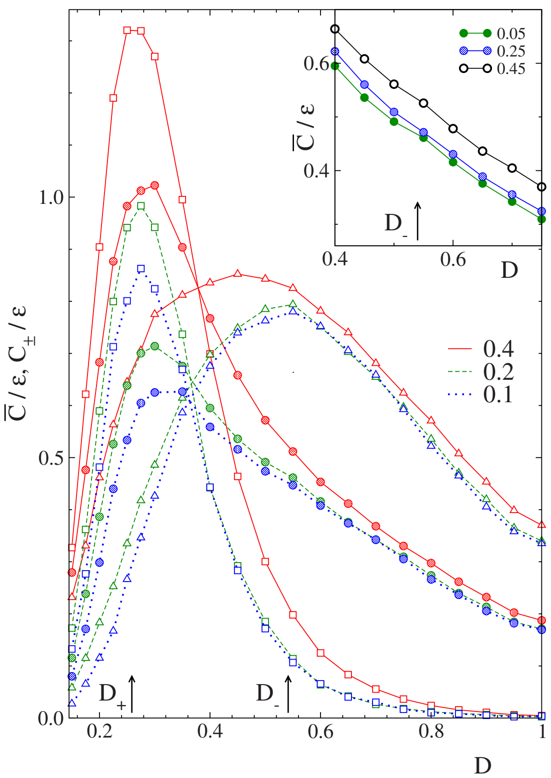


FIG. 3. (Color online) Degree of coherence in the ADW potential (1): $C_+$ (squares), $C_-$ (triangles), and $\bar{C}$ (circles) vs $D$ for different $A$ (in the legend). Vertical arrows locate the peaks, $D_\pm$ of $C_\pm$. Other simulation parameters: $\epsilon \equiv \Delta t / T_\Omega = 0.1$. $\Omega = 0.2$ and the ADW parameters as in Fig. 1 Inset: dependence of the curves $\bar{C}(D)$ on $\epsilon$ and $A$.

tials with the same switch energy barrier, $\Delta V$, in both directions, i.e., in the absence of interwell asymmetry.

At this point, a few important remarks are in order. For the small time bins employed both here, in Fig. 3, and in [9], the quantities $C_\pm$ are linear in $\Delta t$, that is $C_\pm \simeq \Delta t N_\pm(T_\Omega/2)$. Moreover, in the limit of vanishingly small drive amplitudes,

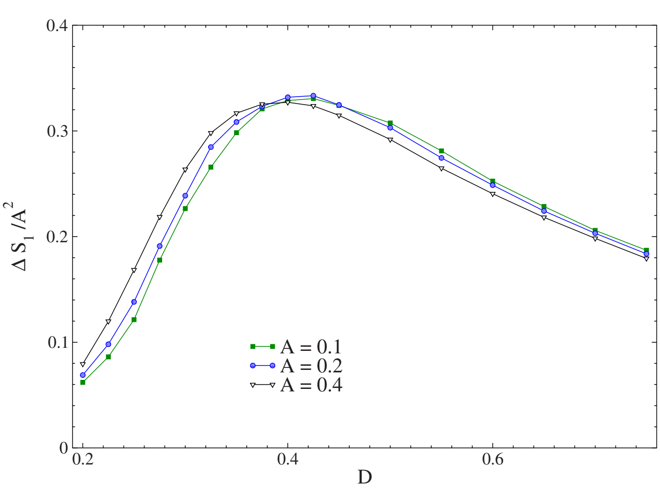


FIG. 4. (Color online) Subtracted strength $\Delta S_1$ of the first spectral peak at $\omega = \Omega$ versus $D$ for different $A$. Other simulation parameters: $\Omega = 0.2$ and the ADW parameters as in Fig. 1

the curves $C_\pm(D)$ and $\bar{C}(D)$ grow insensitive to $A$. Indeed, for zero drives, $N_\pm(T) = \exp(-T/T_K^\pm)/T_K^\pm$ [13] and the functions $C_\pm(D)$ and $\bar{C}(D)$ can be shown analytically to fit our numerical data. In particular, $\bar{C}(D)$ develops a "split" profile even for $A=0$ and arbitrary $T_\Omega$, which only proves the existence of two different Kramers' rates, $T_K^\pm$. Such an artifact of the coherence quantifiers Eq. (5) grows negligible in the nonlinear regime, i.e., for larger time bins, $\Delta t$, or larger drive amplitudes, $A$. However, as shown in the inset of Fig. 3, no prominent double SR $\bar{C}(D)$ peaks were detected under such conditions, either.

In conclusion, our numerical study corroborates earlier observations [6–9] according to which the synchronization conditions over a skewed potential barrier depend on the orientation of the transitions. However, the ensuing "SR splitting" effect appears to be far less significant than reported by Mueller and co-workers [9].

The existence of a double SR has be regarded as a mere synchronization effect. In fact, no splitting of the SR peak was detected in the spectral analysis of the stochastic process $x(t)$. When subjected to a low-frequency periodic drive, the time periodic response of the cyclostationary asymmetric process Eqs. (1) and (2) is characterized by the appearance of deltalike peaks in its p.s.d., $S(\omega)$. Such peaks emerge from the spectral background of the random (or incoherent) switches, and are located at $\omega = n\Omega$, with $n$ an integer [1,6]. On adopting a standard (spectral) characterization of SR [1], in Fig. 4 we plotted the subtracted strength,

$$\Delta S_1 = \lim_{\delta \to 0} \int_{\Omega-\delta}^{\Omega+\delta} [S(\omega) - S_0(\omega)] d\omega, \tag{7}$$

of the most prominent peak with $n=1$ versus the noise intensity $D$. Of course, all p.s.d. have been computed for the same frequency bin, $\delta$, and the background densities, $S_0(\omega)$, were estimated by numerical interpolation.

The numerical curves reported in Fig. 4 do not show the slightest evidence of double SR. This is no surprise as $\Delta S_1(\Omega)$ is known [1,13] to depend on a symmetric combination of the average residence times, $T_K^\pm$, i.e., $T_K^+ T_K^- / (T_K^+ + T_K^-)$. As a consequence, it is not possible to fine-tune $D$ so as to extract information regarding one-sided transitions, only.

## IV. CONCLUDING REMARKS

In the foregoing section, we have given numerical evidence that intrawell asymmetry maybe indeed responsible for the "SR splitting" reported in [9]. Such an effect is a manifestation of asymmetric synchronization over an skewed barrier [14], with no spectral counterpart.

Of course, asymmetry effects are known to emerge also in the spectral characterization of SR. We mentioned in Sec. I that asymmetry causes the appearance of additional p.s.d. peaks at even multiples of the drive frequency, and that the strength of each peak exhibits a peculiar SR dependence on $D$ [6]. In this Brief Report, we simply ruled out the existence of a spectral signature of the double SR effect.

Finally, we also investigated the occurrence of double SR in the presence of interwell asymmetry. Following [6,7], we studied the Langevin Eq. (2) for the case of a tilted symmetric potential, $V(x)=-ax^2/2+bx^4/4-A_0x$, with $a,b$ and $A_0>0$. The corresponding coherence quantifiers $C_\pm$ peak for different noise intensities, with $D_+>D_-$, but their average, $\bar{C}$, shows no trace of "SR splitting." Note that the tilt $A_0$ tends to suppress (increase) the energy barrier opposing the $\mp\rightarrow\pm$ transition and, simultaneously, to make the negative (positive) well shallower (narrower). Such modifications of the initially symmetric potential have opposite bearing on the $D$ dependence the Kramers' times $T_K^\pm$, see Eq. (4). We think that this is the main reason why we were unable to detect double SR in this class of ADW potentials, even if the opposite barrier transitions can be optimally synchronized (i.e., $C_\pm$ maximized) for distinct noise intensities.


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